Moving boundary problems on Earth's surface

Piotr Rybka (University of Warsaw) & Vaughan Voller (University of Minnesota)

In a moving boundary problem one or more of the domain boundaries is an unknown function of time. The classic moving boundary problem, the Stefan problem, is related to the tracking of a sharp liquid/solid front during the melting of ice.

A major societal problem of our time is the current rapid rate of degradation of Earth environments and resources through climate change and other anthropogenic forcing. Finding mitigation strategies and solutions to these problems requires large scale trans-disciplinary research efforts.

A central plank in this research is the need to develop an understanding of the transport processes that govern how materials and resources are moved through and over the Earth's surface. Two relevant examples, which can be seen as generalizations of the classic Stefan problem, are the sub-surface movement of resources such as water and the surface transport of sediment which builds and maintains landscapes and ocean shorelines. A feature in these problems is that the transport domain, i.e., the Earth's surface, is highly heterogeneous exhibiting a wide ranges of length scales that are often power-law distributed. The consequence of this is that the transport processes are non-local in space and time, i.e., fluxes of conserved quantities cannot be determined from instantaneous and local conditions alone. Recently there has been much interest in the geomorphodynamics research community of exploring how fractional calculus representations of elements in the transport equations can be used to model non-locality, [3], [8], [7], [4]. Phenomenological fractional calculus models of geomorphic and geology transport processes that provide sound qualitative comparisons with filed and experimental observations have been constructed. These models have suggested many interesting hypothesis on how the controlling mechanisms of transport processes shape our environment. In this work, however, there is an almost complete absence of rigorous mathematical analysis.

Let us recall the statement of the so-called one-phase Stefan, where u is the temperature of the melting solid and s is position of the interface, ($x = 0$ is the fixed boundary of a container). For the sake of simplicity we consider the one-dimensional case,

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) \quad \text{for } x \in (0, s(t)),\ t > 0, \\ &\frac{ds}{dt} = L\frac{\partial u}{\partial x}\Big|_{x=s(t)} \quad \text{for } t > 0. \end{aligned} \tag{1}$$

The essential part of the surface transport of sediment looks like, see [5]

$$\begin{aligned} &\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) \quad \text{for } x \in (0, s(t)),\ t > 0, \\ &\frac{d^\alpha s}{dt^\alpha} = L\frac{\partial u}{\partial x}\Big|_{x=s(t)} \quad \text{for } t > 0. \end{aligned} \tag{2}$$

Here $\alpha \in (0, 1]$ and $\frac{\partial^\alpha u}{\partial t^\alpha}$ is the fractional time derivative. In this model s is again the moving front position.

In equation (2) we need to emphasis that L can be a function of space and time; the change in space driven by the change in ocean depth the variation in time driven by tectonic processes.

The formal resemblance of these two systems is striking. It is even more interesting to notice that a model of sub-surface movement of water, under the assumption that it moves as a saturated “plug” through a soil that has a constant moisture storage capacity, is obtained, see [6], formally speaking by replacing one space derivative on the RHS of (1) by fractional one, $\frac{\partial^\alpha}{\partial x^\alpha}$,

$$\begin{aligned} &\tfrac{\partial u}{\partial t} = \tfrac{\partial}{\partial x}\left(K \tfrac{\partial^\alpha u}{\partial x^\alpha}\right) \quad \text{for } x \in (0, s(t)),\ t > 0, \\ &\tfrac{ds}{dt} = \tfrac{\partial^\alpha u}{\partial x^\alpha}\Big|_{x=s(t)} \quad \text{for } \ t > 0. \end{aligned} \tag{3}$$

Of course, in each of these models u, x, $s(t)$ denote different physical quantities.

The goal of the project is twofold. One is to develop rigorous theory for a family of problems like (2) or (3). The area of fractional differential equations is emerging, the number of papers on this subject is rather limited and related to Levy stochastic processes, see e.g. [1], or motion of dislocation, see [2]. As a result the free boundary problems offer numerous opportunities to newcomers. Yet, this is not an uncharted territory, because the classical Stefan problem (1) has been extensively studied, giving numerous hints how to attack (2) or (3).

The second goal of this project is developing and deepening the models depending upon understanding the physics. It should be stressed that the one dimensional model above are a gross simplification of two- or three-dimensional descriptions which are the to be created yet. Finally, the computational aspects of (2) or (3) are to be explored.

Contact information Candidates are asked to contact Piotr Rybka at rybka@mimuw.edu.pl prior to qualifying interview.

References

[1] P.Biler, G.Karch, W.A.Woyczyński, Critical nonlinearity exponent and self-similar asymptotics for Lévy conservation laws, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **18** (2001), 613–637.

[2] P.Biler, G.Karch, R.Monneau, Nonlinear diffusion of dislocation density and self-similar solutions, *Comm. Math. Phys.*, **294**, (2010), 145–168.

[3] E.Foufoula-Georgiou, V.Ganti, W.Dietrich, A nonlocal theory of sediment transport on hillslopes, *J. Geophys. Res.*, **115**, (2010), F00A16, doi:10.1029/2009JF001280.

[4] R.Schumer, M.M.Meerschaert, B.Baeumer, Fractional advection-dispersion equations for modeling transport at the Earth surface, *J. Geophys. Res.*, **114**, (2009), F00A07, doi:10.1029/2008JF001246.

[5] V.R. Voller, An exact solution of a limit case Stefan problem governed by a fractional diffusion equation, *International Journal of Heat and Mass Transfer*, **53**, (2010), 5622-5625

[6] V.R. Voller, On a fractional derivative form of the Green–Ampt infiltration model, *Advances in Water Resources*, **34**, (2010), 257-262.

[7] V.R. Voller, C. Paola, Can anomalous diffusion describe depositional fluvial profiles?, *J. Geophys. Res.*, **115**, (2010) F00A13, doi:10.1029/2009JF001278.

[8] V.R.Voller, V. Ganti, C.Paola, E.Foufoula-Georgiou, Does the flow of information in a landscape have direction?, *Geophys. Res. Lett.*, **39**, (2012), L01403, doi:10.1029/2011GL050265.